

# STATIONARITY AND SELF-SIMILARITY CHARACTERIZATION OF THE SET-INDEXED FRACTIONAL BROWNIAN MOTION

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**ABSTRACT.** The set-indexed fractional Brownian motion (sifBm) has been defined by Herbin-Merzbach (2006a) for indices that are subsets of a metric measure space. In this paper, the sifBm is proved to satisfy a strengthened definition of increment stationarity. This new definition for stationarity property allows to get a complete characterization of this process by its fractal properties: The sifBm is the only set-indexed Gaussian process which is self-similar and has stationary increments.

Using the fact that the sifBm is the only set-indexed process whose projection on any increasing path is a one-dimensional fractional Brownian motion, the limitation of its definition for a self-similarity parameter  $0 < H < 1/2$  is studied, as illustrated by some examples. When the indexing collection is totally ordered, the sifBm can be defined for  $0 < H < 1$ .

## 1. INTRODUCTION

In [HeMe06a], the set-indexed fractional Brownian motion (sifBm) was defined among processes indexed by a collection of subsets of a measure metric space. The study of its properties showed fractal behaviour such as a kind of increment stationarity and self-similarity. In addition, it is proved that the projection of a sifBm on an increasing path is a one-parameter time changed fractional Brownian motion. Fine properties of multi-dimensional parameter fractional Brownian motions are studied by several authors (see [Is05, TuXi08, XiZh02] for example).

In this paper, we extend the increment stationarity property defined in [HeMe06a]. Instead of considering a stationarity property on  $\Delta X_C$  (for  $C \in \mathcal{C}_0$ ) that only involves marginal distributions of the increment process, we consider a property of stationarity of the distribution of the whole process  $\Delta X = \{\Delta X_C; C \in \mathcal{C}_0\}$ . We obtain a strengthened definition for increment stationarity which is preserved under projections on flows (increasing paths). More precisely, we show that if  $X$  is a set-indexed process satisfying this new property of stationarity, then its projection on any flow is a one-dimensional increment stationary process. For that reason, this new definition can be considered as the most natural one. The set-indexed fractional Brownian motion is proved to satisfy this property.

The new stationarity definition allows us to get the main result of this paper: a complete characterization of the set-indexed fractional Brownian motion as the only set-indexed mean-zero Gaussian process which satisfies the two properties of increment stationarity and self-similarity. This property thus extends the well-known characterization of one-parameter fractional Brownian motion.

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2000 *Mathematics Subject Classification.* 62G05, 60G15, 60G17, 60G18.

*Key words and phrases.* fractional Brownian motion, Gaussian processes, stationarity, self-similarity, set-indexed processes.

The second point of this paper is the use of flows to understand the limitation of the general sifBm's definition for a parameter  $H \in (0, 1/2]$ , as opposed to one-parameter fractional Brownian motion which is defined for  $0 < H < 1$ . In the latter case, the behaviour of the process leads to critical values for  $H$  (see [Ch01, ChNu05] for instance). Here we observe that the set-indexed fractional Brownian motion can be defined for  $0 < H < 1$  when the indexing collection  $\mathcal{A}$  is totally ordered. On the contrary, we give new examples of indexing collections  $\mathcal{A}$  on which the sifBm cannot be defined for  $H > 1/2$ .

The paper is organized as follows: in section 2, indexing collection and the set-indexed fractional Brownian motion are defined, and projections of the sifBm on flows are studied. Section 3 is devoted to the study of the sifBm along flows to get a deeper understanding of its properties. Among the results, we get a characterization of the sifBm by its projection along flows, which constitutes a converse of a result in [HeMe06a]. We prove that a set-indexed process is a set-indexed fractional Brownian motion if and only if its projections on every flows are one-dimensional fractional Brownian motions. This gives a good justification of the definition of the sifBm and opens the door to a variety of applications. In [HeMe06c], a part of this result was presented and also an integral representation for the sifBm was given.

In section 4, we use the flows to get a better understanding of the limitation  $H \in (0, 1/2]$ . This fact was already observed for the fBm indexed by the sphere of  $\mathbf{R}^N$  (see [Is05]). The examples given explain why the sifBm cannot be defined in general for  $H > 1/2$ . As a byproduct, we prove that the cardinality of a totally ordered indexing collection cannot exceed the continuum.

In section 5, the new strengthened definition for increment stationarity of a set-indexed process is studied.

Then in section 6, we establish the fractal characterization of the set-indexed fractional Brownian motion.

## 2. PROJECTIONS OF THE SIFBM ON FLOWS

We follow [HeMe06a] for the framework and notation. Our processes are indexed by an *indexing collection*  $\mathcal{A}$  of compact subsets of a locally compact metric space  $\mathcal{T}$  equipped with a Radon measure  $m$  (denoted  $(\mathcal{T}, m)$ ).

Let  $\mathcal{A}(u)$  denotes the class of finite unions from sets belonging to  $\mathcal{A}$ .

**Definition 2.1** (Indexing collection). *A nonempty class  $\mathcal{A}$  of compact, connected subsets of  $\mathcal{T}$  is called an indexing collection if it satisfies the following:*

- (1)  $\emptyset \in \mathcal{A}$ , and  $A^\circ \neq A$  if  $A \notin \{\emptyset, \mathcal{T}\}$ . In addition, there exists an increasing sequence  $(B_n)_{n \in \mathbf{N}}$  of sets in  $\mathcal{A}(u)$  such that  $\mathcal{T} = \bigcup_{n \in \mathbf{N}} B_n^\circ$ .
- (2)  $\mathcal{A}$  is closed under arbitrary intersections and if  $A, B \in \mathcal{A}$  are nonempty, then  $A \cap B$  is nonempty. If  $(A_i)$  is an increasing sequence in  $\mathcal{A}$  and if there exists  $n \in \mathbf{N}$  s. t. for all  $i$ ,  $A_i \subseteq B_n$  then  $\overline{\bigcup_i A_i} \in \mathcal{A}$ .
- (3) The  $\sigma$ -algebra generated by  $\mathcal{A}$ ,  $\sigma(\mathcal{A}) = \mathcal{B}$ , the collection of all Borel sets of  $\mathcal{T}$ .
- (4) Separability from above  
*There exists an increasing sequence of finite subclasses  $\mathcal{A}_n = \{A_1^n, \dots, A_{k_n}^n\}$  of  $\mathcal{A}$  closed under intersections and satisfying  $\emptyset, B_n \in \mathcal{A}_n(u)$  and a sequence of functions  $g_n : \mathcal{A} \rightarrow \mathcal{A}_n(u) \cup \{\mathcal{T}\}$  such that*

- (a)  $g_n$  preserves arbitrary intersections and finite unions
    - (i. e.  $g_n(\bigcap_{A \in \mathcal{A}'} A) = \bigcap_{A \in \mathcal{A}'} g_n(A)$  for any  $\mathcal{A}' \subseteq \mathcal{A}$ , and if  $\bigcup_{i=1}^k A_i = \bigcup_{j=1}^m A'_j$ , then  $\bigcup_{i=1}^k g_n(A_i) = \bigcup_{j=1}^m g_n(A'_j)$ );
  - (b) for each  $A \in \mathcal{A}$ ,  $A \subseteq (g_n(A))^\circ$ ;
  - (c)  $g_n(A) \subseteq g_m(A)$  if  $n \geq m$ ;
  - (d) for each  $A \in \mathcal{A}$ ,  $A = \bigcap_n g_n(A)$ ;
  - (e) if  $A, A' \in \mathcal{A}$  then for every  $n$ ,  $g_n(A) \cap A' \in \mathcal{A}$ , and if  $A' \in \mathcal{A}_n$  then  $g_n(A) \cap A' \in \mathcal{A}_n$ ;
  - (f)  $g_n(\emptyset) = \emptyset \forall n$ .
- (5) Every countable intersection of sets in  $\mathcal{A}(u)$  may be expressed as the closure of a countable union of sets in  $\mathcal{A}$ .

(Note: ' $\subset$ ' indicates strict inclusion and ' $\overline{(\cdot)}$ ' and ' $(\cdot)^\circ$ ' denote respectively the closure and the interior of a set.)

The *set-indexed fractional Brownian motion (sifBm)* on  $(\mathcal{T}, \mathcal{A}, m)$  was defined as the centered Gaussian process  $\mathbf{B}^H = \{\mathbf{B}_U^H; U \in \mathcal{A}\}$  such that

$$\forall U, V \in \mathcal{A}; E[\mathbf{B}_U^H \mathbf{B}_V^H] = \frac{1}{2} [m(U)^{2H} + m(V)^{2H} - m(U \triangle V)^{2H}], \quad (1)$$

where  $0 < H \leq \frac{1}{2}$ .

If  $\mathcal{A}$  is provided with a structure of group on  $\mathcal{T}$ , properties of increment stationarity and self-similarity are studied in [HeMe06a]. In the special case of  $\mathcal{A} = \{[0, t]; t \in \mathbf{R}_+\} \cup \{\emptyset\}$ , we get a multiparameter process called Multiparameter fractional Brownian motion (MpfBm), whose properties are studied in [HeMe06b].

The notion of flow is the key to reduce the proof of many theorems. It was extensively studied in [Iv03] and [IvMe00].

**Definition 2.2.** An elementary flow is defined to be a continuous increasing function  $f : [a, b] \subset \mathbf{R}_+ \rightarrow \mathcal{A}$ , i. e. such that

$$\begin{aligned} \forall s, t \in [a, b]; \quad s < t &\Rightarrow f(s) \subseteq f(t) \\ \forall s \in [a, b]; \quad f(s) &= \bigcap_{v > s} f(v) \\ \forall s \in (a, b); \quad f(s) &= \overline{\bigcup_{u < s} f(u)}. \end{aligned}$$

A simple flow is a continuous function  $f : [a, b] \rightarrow \mathcal{A}(u)$  such that there exists a finite sequence  $(t_0, t_1, \dots, t_n)$  with  $a = t_0 < t_1 < \dots < t_n = b$  and elementary flows  $f_i : [t_{i-1}, t_i] \rightarrow \mathcal{A}$  ( $i = 1, \dots, n$ ) such that

$$\forall s \in [t_{i-1}, t_i]; \quad f(s) = f_i(s) \cup \bigcup_{j=1}^{i-1} f_j(t_j).$$

The set of all simple (resp. elementary) flows is denoted by  $S(\mathcal{A})$  (resp.  $S^e(\mathcal{A})$ ).

In [IvMe00], the projection of a set-indexed process  $X = \{X_U; U \in \mathcal{A}\}$  on any elementary flow  $f$  was considered as the real-parameter process  $X^f = \{X_{f(t)}; t \in [a, b] \subset \mathbf{R}_+\}$ . Here, we define another parametrization of this projection, which allows simpler statements in the sequel.

**Definition 2.3.** For any set-indexed process  $X = \{X_U; U \in \mathcal{A}\}$  on the space  $(\mathcal{T}, \mathcal{A}, m)$  and any elementary flow  $f : [a, b] \rightarrow \mathcal{A}$ , we define the  $m$ -standard projection of  $X$  on  $f$  as the process

$$X^{f,m} = \left\{ X_t^{f,m} = X_{f \circ \theta^{-1}(t)}; t \in [a, b] \right\},$$

where  $\theta : t \mapsto m[f(t)]$  and  $\theta^{-1}$  is its pseudo-inverse function.

The use of this new notation  $X^{f,m}$  avoids any confusion with the projection  $X^f$  previously defined.

Notice that since  $\theta$  is non-decreasing, the function  $\theta^{-1}$  is well-defined and for all  $t \in [a, b]$ , we have  $\theta(\theta^{-1}(t)) = t$ .

The following result, proved in [HeMe06a], gives a good justification of the definition of the  $\text{sifBm}$ .

**Proposition 2.4.** Let  $\mathbf{B}^H$  be a  $\text{sifBm}$  on  $(\mathcal{T}, \mathcal{A}, m)$  and  $f : [a, b] \rightarrow \mathcal{A}$  be an elementary flow. Then the process  $(\mathbf{B}^H)^{f,m} = \{\mathbf{B}_{f \circ \theta^{-1}(t)}^H, t \in [a, b]\}$ , where  $\theta : t \mapsto m[f(t)]$ , is a one-parameter fractional Brownian motion.

In section 3, we prove the converse to Proposition 2.4. For this purpose, we will use the following lemma proved in [Iv03].

**Lemma 2.5.** The finite dimensional distributions of an additive  $\mathcal{A}$ -indexed process  $X$  determine and are determined by the finite dimensional distributions of the class  $\{X^f, f \in S(\mathcal{A})\}$ .

### 3. CHARACTERIZATION OF THE $\text{SIFBM}$

In the case of  $L^2$ -monotone outer-continuous set-indexed processes, we prove that the  $\text{sifBm}$  could be defined as a process whose projections on elementary flows is a one-dimensional fractional Brownian motion.

Recall the following definition (see [IvMe00])

**Definition 3.1.** A set-indexed process  $X = \{X_U; U \in \mathcal{A}\}$  is said  $L^2$ -monotone outer-continuous if for any decreasing sequence  $(U_n)_{n \in \mathbf{N}}$  of sets in  $\mathcal{A}$ ,

$$E [|X_{U_n} - X_{\bigcap_{k \in \mathbf{N}} U_k}|^2] \rightarrow 0$$

as  $n \rightarrow \infty$ .

**Proposition 3.2.** The  $\text{sifBm}$   $\mathbf{B}^H$  ( $0 < H \leq 1/2$ ) is  $L^2$ -monotone outer-continuous.

*Proof.* Let  $(U_n)_{n \in \mathbf{N}}$  be a decreasing sequence in  $\mathcal{A}$ . As  $\bigcap_{k \in \mathbf{N}} U_k \in \mathcal{A}$ , by definition of  $\text{sifBm}$ , we have

$$\forall n \in \mathbf{N}; \quad E \left[ |\mathbf{B}_{U_n}^H - \mathbf{B}_{\bigcap_{k \in \mathbf{N}} U_k}^H|^2 \right] = m(U_n \setminus \bigcap_{k \in \mathbf{N}} U_k)^{2H}$$

But, as  $(U_n)_{n \in \mathbf{N}}$  is decreasing, by definition of a measure,

$$m(U_n \setminus \bigcap_{k \in \mathbf{N}} U_k) \rightarrow 0.$$

Then the result follows.  $\square$

The following lemma will be useful for the converse of proposition 2.4, and will be strengthened in section 4 to understand links between structure of  $\mathcal{A}$  and flows.

**Lemma 3.3.** *For any  $U_1, U_2, \dots, U_n \in \mathcal{A}$  such that  $U_i \subset U_{i+1}$  ( $\forall i = 1, \dots, n-1$ ), there exist an elementary flow  $f : \mathbf{R}_+ \rightarrow \mathcal{A}$  and real numbers  $0 < t_1 < t_2 < \dots < t_n$  such that*

$$\forall i = 1, \dots, n; \quad f(t_i) = U_i.$$

*Proof.* This result is a particular case of lemma 5.1.7 in [IvMe00] (and lemma 5 in [Iv03]). As the sequence  $U_1, U_2, \dots, U_n$  is increasing,  $\mathcal{A}' = \{U_1, U_2, \dots, U_n\}$  constitutes a semilattice of  $\mathcal{A}$  with a consistent numbering. The proof of lemma 5.1.7 in [IvMe00] constructs such an elementary flow  $f$ . Here the increasing property of  $(U_i)_{1 \leq i \leq n}$  allows  $f$  to take its values in  $\mathcal{A}$  ( $\subset \mathcal{A}(u)$ ).  $\square$

**Theorem 3.4.** *Let  $X = \{X_U; U \in \mathcal{A}\}$  be an  $L^2$ -monotone outer-continuous set-indexed process.*

*If the projection  $X^f$  of  $X$  on any elementary flow  $f$ , is a time-changed one-parameter fractional Brownian motion of parameter  $H \in (0, 1/2]$ , then there exists a Borel measure  $\nu$  on  $\mathcal{T}$  such that  $X$  is a set-indexed fractional Brownian motion on  $(\mathcal{T}, \mathcal{A}, \nu)$ .*

This theorem states that the time-changes giving to projections the law of a one-parameter fBm, determine a Borel measure  $\nu$  such that  $X$  is a sifBm on the space  $(\mathcal{T}, \mathcal{A}, \nu)$ .

A sketch of the proof is given in [HeMe06c]. Here we present a complete proof. In particular, the importance of lemma 3.3 is pointed out.

*Proof.* Let  $f : [a, b] \rightarrow \mathcal{A}$  be an elementary flow. As the projected process  $X^f$  is a time-changed fBm of parameter  $H$ , we have

$$\forall s, t \in [a, b]; \quad E \left[ X_t^f - X_s^f \right]^2 = |\theta_f(t) - \theta_f(s)|^{2H} \quad (2)$$

where  $\theta_f : \mathbf{R}_+ \rightarrow \mathbf{R}_+$  is an increasing function.

The idea of the proof is the construction of a measure  $\nu$  such that for any  $f \in S^e(\mathcal{A})$ ,

$$\forall t \in [a, b]; \quad \theta_f(t) = \nu[f(t)].$$

For all  $U \in \mathcal{A}$ , let us define

$$F_U^e = \{f \in S^e(\mathcal{A}) : \exists u_f \in [a, b]; U = f(u_f)\}.$$

As for all  $f$  and  $g$  in  $F_U^e$ ,  $\theta_f(u_f)^{2H} = \theta_g(u_g)^{2H} = E[X_U]^2$ , one can define

$$\psi(U) = \theta_f(u_f) = (E[X_U]^2)^{\frac{1}{2H}}. \quad (3)$$

For all  $U$  and  $V$  in  $\mathcal{A}$  with  $U \subset V$ , lemma 3.3 implies the existence of an elementary flow  $f$  such that

$$\exists u_f, v_f \in [a, b]; \quad u_f \leq v_f; \quad U = f(u_f) \subset f(v_f) = V$$

Then, as the time-change  $\theta_f$  is increasing,  $\psi$  is non-decreasing in  $\mathcal{A}$ .

The definition of  $\psi$  on  $\mathcal{A}$  can be extended to  $C = U \setminus \bigcup_{1 \leq i \leq n} U_i$  by the inclusion-exclusion formula

$$\begin{aligned} \psi(C) = \psi(U) - \sum_{i=1}^n \psi(U \cap U_i) + \sum_{i < j} \psi(U \cap (U_i \cap U_j)) \\ - \cdots + (-1)^n \psi \left( U \cap \left( \bigcap_{1 \leq i \leq n} U_i \right) \right). \end{aligned} \quad (4)$$

We denote this class of sets by  $\mathcal{C}$ .

The definition (4) of  $\psi$  can be easily extended to the set  $\mathcal{C}(u)$  of finite unions of elements of  $\mathcal{C}$  in the same way.

A direct consequence of definition (4) is that, for all  $C_1, C_2 \in \mathcal{C}$  such that  $C = C_1 \cup C_2 \in \mathcal{C}$ ,

$$\psi(C_1 \cup C_2) = \psi(C_1) + \psi(C_2) - \psi(C_1 \cap C_2) \quad (5)$$

Let us remark that equality (5) holds for any  $C_1, C_2 \in \mathcal{C}(u)$ .

From the pre-measure  $\psi$  defined on  $\mathcal{C}$ , the function

$$\nu : E \subset \mathcal{T} \mapsto \inf_{\substack{C_i \in \mathcal{C} \\ E \subset \bigcup C_i}} \sum_{i=1}^{\infty} \psi(C_i) \quad (6)$$

defines an outer measure on  $\mathcal{T}$  (see [Ro70] pp. 9–26). Let us show that  $\nu$  defines a Borel measure on the topological space  $\mathcal{T}$ .

Let  $\mathcal{M}_\nu$  be the  $\sigma$ -field of  $\nu$ -measurable subsets of  $\mathcal{T}$ . It is known that  $\nu$  is a measure on  $\mathcal{M}_\nu$  (see [Ro70], thm. 3). By definition, any  $U \in \mathcal{A}$  is  $\nu$ -measurable if

$$\forall A \subset U, \forall B \subset \mathcal{T} \setminus U; \quad \nu(A \cup B) = \nu(A) + \nu(B)$$

As the inequality  $\nu(A \cup B) \leq \nu(A) + \nu(B)$  follows from definition of any outer-measure, it remains to show the converse inequality.

Consider any sequence  $(C_i)_{i \in \mathbb{N}}$  in  $\mathcal{C}$  such that  $A \cup B \subset \bigcup_i C_i$ . The sequence  $(C_i)_{i \in \mathbb{N}}$  can be decomposed in the elements  $C_i$ ,  $i \in I$  such that  $C_i \cap U = \emptyset$  and the  $C_i$ ,  $i \in J$  such that  $C_i \subset U$  (if  $C_i \cap U \neq \emptyset$  and  $C_i \not\subset U$ , cut  $C_i = C'_i \cup C''_i$  where  $C'_i \subset U$  and  $C''_i \cap U = \emptyset$ ).

As

$$A \cup B \subset \left[ \bigcup_{i \in I} C_i \right] \cup \left[ \bigcup_{i \in J} C_i \right]$$

we get the implications

$$\forall i \in I; C_i \cap U = \emptyset \quad \Rightarrow \quad A \subset \bigcup_{i \in J} C_i$$

and

$$\forall i \in J; C_i \subset U \quad \Rightarrow \quad B \subset \bigcup_{i \in I} C_i.$$

Then,

$$\sum_{i=1}^{\infty} \psi(C_i) = \underbrace{\sum_{i \in I} \psi(C_i)}_{\geq \nu(B)} + \underbrace{\sum_{i \in J} \psi(C_i)}_{\geq \nu(A)}$$

which leads to  $\nu(A \cup B) \geq \nu(A) + \nu(B)$ .

We have proved that  $\mathcal{A} \subset \mathcal{M}_\nu$ . By definition of  $\mathcal{A}$ , the smallest  $\sigma$ -field containing  $\mathcal{A}$  is the Borel  $\sigma$ -field  $\mathcal{B}$ . Therefore,  $\mathcal{B} \subset \mathcal{M}_\nu$  and  $\nu$  is a measure on  $\mathcal{B}$ .

The second part of the proof is to show that the measure  $\nu$  is an extension of  $\psi$ , i. e.

$$\forall U \in \mathcal{A}; \quad \nu(U) = \psi(U). \quad (7)$$

- For any  $U \in \mathcal{A}$ , by definition of  $\nu(U)$ ,

$$\nu(U) = \inf_{\substack{C_i \in \mathcal{C} \\ U \subset \bigcup C_i}} \sum_{i=1}^{\infty} \psi(C_i) \leq \psi(U) \quad (8)$$

- To prove the converse inequality, consider  $U \in \mathcal{A}$  and a sequence  $(C_i)_{i \in \mathbb{N}}$  in  $\mathcal{C}$  such that  $U \subset \bigcup_i C_i$ . For all  $n \in \mathbb{N}^*$ , we have

$$U \subset \bigcup_{1 \leq i \leq n} C_i \cup \left[ U \setminus \bigcup_{1 \leq i \leq n} C_i \right].$$

Then, (5) implies

$$\begin{aligned} \psi(U) &= \psi \left( \bigcup_{1 \leq i \leq n} C_i \right) + \psi \left( U \setminus \bigcup_{1 \leq i \leq n} C_i \right) \\ &\leq \sum_{i=1}^{\infty} \psi(C_i) + \psi \left( U \setminus \bigcup_{1 \leq i \leq n} C_i \right). \end{aligned} \quad (9)$$

Using  $L^2$ -monotone outer continuity of  $X$  and proposition 1.4.8 in [IvMe00], we have

$$\lim_{n \rightarrow \infty} \psi \left( U \setminus \bigcup_{1 \leq i \leq n} C_i \right) = 0 \quad (10)$$

Thus, (9) and (10) imply that for all sequence  $(C_i)_{i \in \mathbb{N}}$  in  $\mathcal{C}$  such that  $U \subset \bigcup_i C_i$ ,

$$\psi(U) \leq \sum_{i=1}^{\infty} \psi(C_i)$$

and then, by definition of  $\nu(U)$

$$\psi(U) \leq \nu(U) \quad (11)$$

Equality (7) follows from (8) and (11).

From (3) and (7), the Borel measure  $\nu$  defined by (6) satisfies

$$\forall U \in \mathcal{A}; \quad E[X_U]^2 = \psi(U)^{2H} = \nu(U)^{2H}.$$

Using the Borel measure  $\nu$ , consider a set-indexed fractional Brownian motion  $Y$  on  $(\mathcal{T}, \mathcal{A}, \nu)$  (which exists as  $0 < H \leq 1/2$ ), defined by

$$\forall U, V \in \mathcal{A}; \quad E[Y_U Y_V] = \frac{1}{2} [\nu(U)^{2H} + \nu(V)^{2H} - \nu(U \triangle V)^{2H}].$$



According to proposition 6.4 in [HeMe06a], projections of  $Y$  on any elementary flow  $f : [a, b] \rightarrow \mathcal{A}$  is a time-change one-parameter fractional Brownian motion, i. e. such that

$$\begin{aligned} \forall s, t \in [a, b]; \quad E \left[ Y_t^f - Y_s^f \right]^2 &= |\nu[f(t)] - \nu[f(s)]|^{2H} \\ &= |\theta_f(t) - \theta_f(s)|^{2H} \end{aligned}$$

Then, the projections of the set-indexed processes  $X$  and  $Y$  on any elementary flow have the same distribution. By additivity, this fact holds also on any simple flow. Thus, lemma 2.5 implies  $X$  and  $Y$  have the same law.  $\square$

Considering only  $m$ -standard projections on flows, theorem 3.4 gives the following characterization of the sifBm.

**Corollary 3.5.** *Let  $X = \{X_U; U \in \mathcal{A}\}$  be an  $L^2$ -monotone outer-continuous set-indexed process. The following two assertions are equivalent:*

- (i) *for any elementary flow  $f : [a, b] \rightarrow \mathcal{A}$ , the  $m$ -standard projection of  $X$  on  $f$  is a one-parameter fractional Brownian motion of index  $H \in (0, 1/2]$ ;*
- (ii)  *$X$  is a set-indexed fractional Brownian motion of index  $H \in (0, 1/2]$  on  $(\mathcal{T}, \mathcal{A}, m)$ .*

#### 4. CAN SIFBM BE DEFINED FOR $H > 1/2$ ?

In [HeMe06a], the set-indexed fractional Brownian motion is defined for a parameter  $H \in (0, 1/2]$ . As one-dimensional fractional Brownian motion is defined for  $H \in (0, 1)$ , a natural question arises: Are there conditions on the indexing collection  $\mathcal{A}$  such that sifBm on  $(\mathcal{T}, \mathcal{A}, m)$  can be defined for  $H > 1/2$ ? Projections on flows allow to answer this question.

Let  $\Phi^H : \mathcal{A} \times \mathcal{A} \rightarrow \mathbf{R}$  denote the function

$$\Phi^H : (U, V) \mapsto \frac{1}{2} \{m(U)^{2H} + m(V)^{2H} - m(U \triangle V)^{2H}\}.$$

The question is: In which cases  $\Phi^H$  can be seen as the covariance function of a set-indexed process? In the following, we can see that this question is related to the two different cases either  $\mathcal{A}$  is totally ordered or not.

Let us first describe the particular structure of a totally ordered indexing collection.

**Proposition 4.1.** *If the indexing collection  $\mathcal{A}$  is totally ordered by the inclusion, then there exists a surjective elementary flow  $f : \mathbf{R}_+ \rightarrow \mathcal{A}$ , i. e. such that*

$$\forall U \in \mathcal{A}; \quad U \in f(\mathbf{R}_+).$$

*Proof.* By definition of an indexing collection,  $\mathcal{A}$  can be discretized by the increasing sequence of finite subclasses  $(\mathcal{A}_n)_{n \in \mathbf{N}}$ . As subclasses  $\mathcal{A}_n$  are finite and totally ordered, lemma 3.3 implies for all  $n$ , existence of an elementary flow  $f_n : \mathbf{R}_+ \rightarrow \mathcal{A}$  such that

$$\mathcal{A}_n \subseteq f_n(\mathbf{R}_+).$$

Note that, by construction of flows  $(f_n)$  (see [IvMe00], lemma 5.1.7), we have

$$\forall t \in f_n^{-1}(\mathcal{A}_n), \forall m \geq n; \quad f_m(t) = f_n(t). \tag{12}$$

Let us define  $\mathcal{I}$ , the set of  $s \in \mathbf{R}_+$  such that  $f_m(s) \in \mathcal{A}_m$  for some  $m \in \mathbf{N}$ .

From the sequence  $(f_n)_{n \in \mathbf{N}}$ , we define the function  $f : \mathbf{R}_+ \rightarrow \mathcal{A}$  in the following way:



- For all  $t \in \mathcal{I}$ , there exists  $m \in \mathbf{N}$  such that  $f_m(t) \in \mathcal{A}_m$ . By (12), the sequence  $(f_n(t))_{n \geq m}$  is constant. We can set  $f(t) = f_m(t)$ .
- In the construction of lemma 5.1.6 in [IvMe00], the subset  $\mathcal{I}$  is proved to be dense. Let us define for all  $t \notin \mathcal{I}$ ,

$$f(t) = \bigcap_{\substack{s \in \mathcal{I} \\ s > t}} f(s).$$

Let us show that  $f$  satisfies the conclusions of the proposition 4.1.

- $f$  is non-decreasing: for all  $s, t \in \mathbf{R}_+$  such that  $s < t$ , we have clearly

$$\bigcap_{\substack{u \in \mathcal{I} \\ u > s}} f(u) \subseteq \bigcap_{\substack{u \in \mathcal{I} \\ u > t}} f(u)$$

and then,  $f(s) \subseteq f(t)$ .

- $f$  passes through every elements of  $\bigcup_{n \in \mathbf{N}} \mathcal{A}_n$ : for any  $U \in \bigcup_{n \in \mathbf{N}} \mathcal{A}_n$ , there exist  $m \in \mathbf{N}$  and  $t_U \in \mathcal{I}$  such that  $U = f_m(t_U)$ . Then, by definition of  $f$  on  $\mathcal{I}$ , we have  $f(t_U) = U$ .
- $f$  is continuous: according to definition 2.2, we must verify that  $f(t) = \bigcap_{s > t} f(s)$  and  $f(t) = \overline{\bigcup_{s < t} f(s)}$ . Using density of  $\mathcal{I}$ , the right-continuity of  $f$  comes directly from its definition

$$f(t) = \bigcap_{\substack{s \in \mathcal{I} \\ s > t}} f(s) = \bigcap_{s > t} f(s).$$

For the second equality, in the proof of lemma 5.1.6 in [IvMe00], it is proved that

$$\overline{\bigcup_{\substack{s \in \mathcal{I} \\ s < t}} f(s)} = \bigcap_{\substack{s \in \mathcal{I} \\ s > t}} f(s).$$

Then the density of  $\mathcal{I}$  allows to conclude  $f(t) = \overline{\bigcup_{s < t} f(s)}$ .

- $f$  passes through all the elements of  $\mathcal{A}$ : for all  $U \in \mathcal{A} \setminus \bigcup_n \mathcal{A}_n$ , its approximations satisfy

$$\forall n \in \mathbf{N}; \quad g_n(U) \in \mathcal{A}_n \text{ and } g_n(U) = \mathcal{A}_n$$

because  $\mathcal{A}_n$  is totally ordered for all  $n$ . Therefore, for all  $n \in \mathbf{N}$ , there exists  $t_{g_n(U)}$  in  $\mathbf{R}_+$  such that  $g_n(U) = f_n(t_{g_n(U)})$ , and we can write

$$U = \bigcap_{n \in \mathbf{N}} g_n(U) = \bigcap_{n \in \mathbf{N}} f_n(t_{g_n(U)}) = \bigcap_{n \in \mathbf{N}} f(t_{g_n(U)}).$$

The sequence  $(t_{g_n(U)})_{n \in \mathbf{N}}$  is non-increasing in  $\mathbf{R}_+$ : By definition,

$$\begin{aligned} g_{n+1}(U) &= f_{n+1}(t_{g_{n+1}(U)}) \\ g_n(U) &= f_n(t_{g_n(U)}) = f_{n+1}(t_{g_n(U)}) \end{aligned}$$

using the construction of  $(f_n)_{n \in \mathbf{N}}$ . Then, as  $(g_n(U))_{n \in \mathbf{N}}$  is non-increasing and  $f_{n+1}$  is non-decreasing, we get  $t_{g_{n+1}(U)} \leq t_{g_n(U)}$  from  $g_n(U) \subseteq g_{n+1}(U)$ .

Consequently,  $(t_{g_n(U)})_{n \in \mathbf{N}}$  converges to some  $t_U \in \mathbf{R}_+$ . Then, using the continuity of  $f$ , we get

$$\bigcap_{n \in \mathbf{N}} f(t_{g_n(U)}) = f(t_U)$$

which proves that  $U \in f(\mathbf{R}_+)$ .

□

In [IvMe00], proposition 1.3.5 shows that by definition, an indexing collection is not allowed to be too big. More precisely, the cardinality of  $\mathcal{A}$  cannot exceed the cardinality of  $\mathcal{P}(\mathbf{R})$ , the set of subsets of  $\mathbf{R}$ . In the particular case of a totally ordered indexing collection, this upper bound for size of  $\mathcal{A}$  can be sharpened. From surjectivity of the flow  $f$  in proposition 4.1, we can state

**Corollary 4.2.** *If the indexing collection  $\mathcal{A}$  is totally ordered by the inclusion, then its cardinality cannot exceed cardinality of  $\mathbf{R}$ .*

From this study of the particular case of a totally ordered collection  $\mathcal{A}$ , we can prove the existence the set-indexed fractional Brownian motion for a parameter  $H \in (0, 1)$ , as in one-parameter case.

**Theorem 4.3.** *If the indexing collection  $\mathcal{A}$  is totally ordered by the inclusion, then the set-indexed fractional Brownian motion on  $(\mathcal{T}, \mathcal{A}, m)$  can be defined for  $0 < H < 1$ .*

*Proof.* According to proposition 4.1, as  $\mathcal{A}$  is totally ordered, there exists an elementary flow  $f : \mathbf{R}_+ \rightarrow \mathcal{A}$  passing through every  $U \in \mathcal{A}$ , i. e. such that

$$\forall U \in \mathcal{A}; \quad U \in f(\mathbf{R}_+).$$

Then, the existence of the sifBm is equivalent to the existence of its projection on the flow  $f$ . For any  $H \in (0, 1)$ , let us consider a one-parameter fractional Brownian motion  $B^H = \{B_t^H; t \in \mathbf{R}_+\}$  of self-similarity parameter  $H$ .

The set-indexed process  $X = \{X_U = B_{\theta \circ f^{-1}(U)}^H; U \in \mathcal{A}\}$ , where  $\theta : t \mapsto m[f(t)]$ , is a mean-zero Gaussian process such that for all  $U, V \in \mathcal{A}$ ,

$$\begin{aligned} E[X_U X_V] &= \frac{1}{2} [(\theta \circ f^{-1}(U))^{2H} + (\theta \circ f^{-1}(V))^{2H} - |\theta \circ f^{-1}(U) - \theta \circ f^{-1}(V)|^{2H}] \\ &= \frac{1}{2} [m(U)^{2H} + m(V)^{2H} - |m(U) - m(V)|^{2H}]. \end{aligned}$$

As  $\mathcal{A}$  is totally ordered, we have either  $U \subseteq V$  or  $V \subseteq U$ , and then

$$|m(U) - m(V)| = m(U \triangle V).$$

Thus, the covariance structure of  $X$  is given by

$$\forall U, V \in \mathcal{A}; \quad E[X_U X_V] = \frac{1}{2} [m(U)^{2H} + m(V)^{2H} - m(U \triangle V)^{2H}]$$

and it follows that  $X$  is a sifBm of parameter  $H$  on  $(\mathcal{T}, \mathcal{A}, m)$ . □

Theorem 4.3 suggests that if there exists a real limitation of sifBm's definition to  $H < 1/2$ , it must be only for partially ordered indexing collections. The following example shows that even in the simple case of rectangles in  $\mathbf{R}^2$ , the sifBm may not be defined for  $H > 1/2$ .

**Example 4.4.** *Let us consider the indexing collection constituted by rectangles of  $\mathbf{R}_+^2$ ,  $\mathcal{A} = \{[0, t]; t \in \mathbf{R}_+^2\} \cup \{\emptyset\}$ .  $\mathcal{A}$  is not totally ordered, and then, theorem 4.3 cannot be applied. In fact, for  $H > 1/2$  the function  $\Phi^H$  is not positive definite.*

*Let us consider the four points of  $\mathbf{R}_+^2$ , defined by their coordinates  $t_1 = (1, 1)$ ,  $t_2 = (2, 1)$ ,  $t_3 = (1, 2)$  and  $t_4 = (2, 2)$ . The points  $t_i$  ( $1 \leq i \leq 4$ ) define four elements  $U_i = [0, t_i]$  of  $\mathcal{A}$ . We compute  $\Phi^H(U_i, U_j)$  for all  $1 \leq i, j \leq 4$  ( $m$  is the Lebesgue measure in  $\mathbf{R}^2$ ). The diagonal terms are*

$$\begin{aligned}\Phi^H(U_1, U_1) &= m(U_1)^{2H} = 1; & \Phi^H(U_2, U_2) &= m(U_2)^{2H} = 2^{2H}; \\ \Phi^H(U_3, U_3) &= m(U_3)^{2H} = 2^{2H}; & \Phi^H(U_4, U_4) &= m(U_4)^{2H} = 4^{2H}.\end{aligned}$$

*The cross terms are*

$$\begin{aligned}\Phi^H(U_1, U_2) &= \frac{1}{2} [m(U_1)^{2H} + m(U_2)^{2H} - m(U_2 \setminus U_1)^{2H}] \\ &= \frac{1}{2} [1 + 2^{2H} - 1] = 2^{2H-1}; \\ \Phi^H(U_1, U_3) &= 2^{2H-1}; \\ \Phi^H(U_1, U_4) &= \frac{1}{2} [1 + 4^{2H} - 3^{2H}];\end{aligned}$$

*and*

$$\begin{aligned}\Phi^H(U_2, U_3) &= \frac{1}{2} [2^{2H} + 2^{2H} - 2^{2H}] = 2^{2H-1}; \\ \Phi^H(U_2, U_4) &= \frac{1}{2} [2^{2H} + 4^{2H} - 2^{2H}] = 2^{4H-1}; \\ \Phi^H(U_3, U_4) &= 2^{4H-1}.\end{aligned}$$

*By computation, the matrix*

$$\begin{pmatrix} 1 & 2^{2H-1} & 2^{2H-1} & \frac{1+2^{4H}-3^{2H}}{2} \\ 2^{2H-1} & 2^{2H} & 2^{2H-1} & 2^{4H-1} \\ 2^{2H-1} & 2^{2H-1} & 2^{2H} & 2^{4H-1} \\ \frac{1+2^{4H}-3^{2H}}{2} & 2^{4H-1} & 2^{4H-1} & 2^{4H} \end{pmatrix}$$

*is not positive definite for  $H = 3/4$  (although it is for  $H = 1/2$ ). Therefore  $\Phi^{3/4}$  is not positive definite and consequently, the sifBm cannot be defined on  $(\mathbf{R}_+^2, \mathcal{A}, m)$  for  $H = 3/4$ .*

The following example shows that sifBm's definition can be used to obtain an extension of fractional Brownian motion indexed by a differential manifold. In that case, the choice of the indexing collection on the manifold is crucial and can lead to different processes, whose definitions are limited to  $0 < H \leq 1/2$  or not.

**Example 4.5.** *Suppose we aim to extend fractional Brownian motion for indices in the unit circle  $\mathbb{S}_1$  in  $\mathbf{R}^2$ . Let us fix a point  $O \in \mathbb{S}_1$  and define  $\mathcal{A}$  as the collection*

$\left\{ \widehat{0M}; M \in \mathbb{S}_1 \right\} \cup \{\emptyset\}$ , where  $\widehat{0M}$  denotes the positive oriented arc from  $O$  to  $M$ .  $\mathcal{A}$  is clearly an indexing collection which is totally ordered. Then, theorem 4.3 implies the existence of a *sifBm* on  $(\mathbb{S}_1, \mathcal{A}, m)$  for a parameter  $H \in (0, 1)$ , where  $m$  denotes the Lebesgue measure on  $\mathbb{S}_1$ . It is defined as the mean-zero Gaussian process  $X = \{X_M; M \in \mathbb{S}_1\}$  such that

$$\forall M, M' \in \mathbb{S}_1; \quad E[X_M - X_{M'}]^2 = m(\widehat{0M} \triangle \widehat{0M'})^{2H} = m(\widehat{MM'})^{2H}.$$

Another choice of indexing collection is  $\mathcal{A}' = \left\{ \widehat{0M}; M \in \mathbb{S}_1 \right\} \cup \{\emptyset\}$ , where  $\widehat{0M}$  denotes the smallest arc of circle from  $O$  to  $M$ . As  $\mathcal{A}'$  is not totally ordered, there is a priori a limitation of *sifBm*'s definition on  $(\mathbb{S}_1, \mathcal{A}', m)$  for a parameter  $H \in (0, 1/2)$ .

Another point of view is followed in Istas' extension of fractional Brownian motion indexed by points on the unit circle, considered as a metric space (and not as a measure space). In [Is05], the periodical fractional Brownian motion (PFBM) is defined as the mean-zero Gaussian process  $X = \{X_M; M \in \mathbb{S}_1\}$  such that  $X_O = 0$  (for some  $O \in \mathbb{S}_1$ ) almost surely and

$$\forall M, M' \in \mathbb{S}_1; \quad E[X_M - X_{M'}]^2 = [d(M, M')]^{2H}$$

where  $d(M, M')$  is the distance between  $M$  and  $M'$  on the circle  $\mathbb{S}_1$ .

This process is different from the two previous definitions based on set-indexed fractional Brownian motion, in the sense that the covariance function cannot be expressed in terms of some measure on  $\mathbb{S}_1$ .

However, if we only consider the positive half-circle  $\frac{1}{2}\mathbb{S}_1$  starting from  $O \in \mathbb{S}_1$ , then

$$\forall M, M' \in \frac{1}{2}\mathbb{S}_1; \quad m(\widehat{0M} \triangle \widehat{0M'}) = m(\widehat{MM'})^{2H} = [d(M, M')]^{2H}$$

and the three covariance functions are identical. Therefore the three different processes defined on  $\mathbb{S}_1$  coincide on  $\frac{1}{2}\mathbb{S}_1$ . In that sense, Istas' PFBM on the half-circle can be seen as a particular case of the *sifBm*. Consequently, fractal properties such as stationarity and self-similarity are satisfied by this process on  $\frac{1}{2}\mathbb{S}_1$  (cf. section 5) but stationarity cannot hold on the whole unit circle.

Moreover, as seen later, the characterization by fractal properties leads naturally to our first definition (cf. section 6).

## 5. FRACTAL PROPERTIES

**5.1. Increment stationarity.** The increments of a set-indexed process are defined from the collection of subsets  $\mathcal{C}$ .

For all  $C = U \setminus \bigcup_{1 \leq i \leq n} U_i$ , we define the increment of the process  $X$  on  $C$  by

$$\Delta X_C = X_U - \sum_{i=1}^n X_{U \cap U_i} + \sum_{i < j} X_{U \cap (U_i \cap U_j)} - \cdots + (-1)^n X_{U \cap (\bigcap_{1 \leq i \leq n} U_i)}. \quad (13)$$

This increment is always well defined for the *sifBm*  $\mathbf{B}^H$  since for all  $U, V \in \mathcal{A}$  such that  $U \cup V \in \mathcal{A}$ , we have  $E[X_U + X_V - X_{U \cap V} - X_{U \cup V}]^2 = 0$ .

In [HeMe06a], we defined a stationarity property for a set-indexed process  $X = \{X_U; U \in \mathcal{A}\}$  on  $(\mathcal{T}, \mathcal{A}, m)$  by

$$\forall C, C' \in \mathcal{C}_0; \quad m(C) = m(C') \Rightarrow \Delta X_C \stackrel{(d)}{=} \Delta X_{C'} \quad (14)$$

where  $\mathcal{C}_0$  denotes the set of elements  $U \setminus V$  with  $U, V \in \mathcal{A}$ . Property (14) is called  $\mathcal{C}_0$ -stationarity.

As  $\mathcal{C}_0$ -stationarity only concerns marginal distributions of the increment process  $\Delta X$ , but not distribution of the process, the property is weaker than the classical increment stationarity property for one-parameter processes.

In that view, it seems judicious to strengthen the increment stationarity definition in such a way that projections of a increment stationary set-indexed process on any flow give increment stationary one-parameter processes.

**Definition 5.1.** *A set-indexed process  $X = \{X_U; U \in \mathcal{A}\}$  is said to have  $m$ -stationary  $\mathcal{C}_0$ -increments if for any integer  $n$ , for all  $V \in \mathcal{A}$  and for all increasing sequences  $(U_i)_{1 \leq i \leq n}$  and  $(A_i)_{1 \leq i \leq n}$  in  $\mathcal{A}$ ,*

$$\forall i, m(U_i \setminus V) = m(A_i) \quad \Rightarrow \quad (\Delta X_{U_1 \setminus V}, \dots, \Delta X_{U_n \setminus V}) \stackrel{(d)}{=} (\Delta X_{A_1}, \dots, \Delta X_{A_n}).$$

**Proposition 5.2.** *The sifBm has  $m$ -stationary  $\mathcal{C}_0$ -increments.*

*Proof.* Let  $X = \{X_U; U \in \mathcal{A}\}$  be a sifBm. For any integer  $n$ , let us consider  $V \in \mathcal{A}$  and increasing sequences  $(U_i)_{1 \leq i \leq n}$  and  $(A_i)_{1 \leq i \leq n}$  in  $\mathcal{A}$  such that  $m(U_i \setminus V) = m(A_i)$  ( $\forall 1 \leq i \leq n$ ). Without loss of generality, we can assume  $V \subset U_i$  ( $\forall i$ ). Let us compute for all  $\lambda_1, \dots, \lambda_n$  in  $\mathbf{R}$ ,

$$\begin{aligned} E [\lambda_1 \Delta X_{U_1 \setminus V} + \dots + \lambda_n \Delta X_{U_n \setminus V}]^2 &= \sum_{i,j} \lambda_i \lambda_j E [\Delta X_{U_i \setminus V} \Delta X_{U_j \setminus V}] \\ &= \sum_{i,j} \lambda_i \lambda_j E [(X_{U_i} - X_V)(X_{U_j} - X_V)] \\ &= \sum_{i,j} \lambda_i \lambda_j (E [X_{U_i} X_{U_j}] - E [X_{U_i} X_V] - E [X_V X_{U_j}] + E [X_V]^2) \\ &= \sum_{i,j} \lambda_i \lambda_j (m(U_i \triangle U_j)^{2H} + m(U_i \triangle V)^{2H} + m(U_j \triangle V)^{2H} - m(U_i \triangle U_j)^{2H}). \end{aligned}$$

Assumptions on  $(U_i)_i$  and  $(A_i)_i$  imply  $m(U_i \triangle V) = m(U_i \setminus V) = m(A_i)$ , and as  $(U_i)_i$  is increasing,  $m(U_i \triangle U_j) = |m(U_i \setminus V) - m(U_j \setminus V)| = |m(A_i) - m(A_j)|$ . Then, for all  $\lambda_1, \dots, \lambda_n$  in  $\mathbf{R}$

$$E [\lambda_1 \Delta X_{U_1 \setminus V} + \dots + \lambda_n \Delta X_{U_n \setminus V}]^2 = E [\lambda_1 \Delta X_{A_1} + \dots + \lambda_n \Delta X_{A_n}]^2$$

and, as the process  $\Delta X$  is centered Gaussian, the result follows.  $\square$

**Example 5.3.** *Following the notation of example 4.5, the sifBm defined on  $(\mathbb{S}_1, \mathcal{A}, m)$ , which provides an extension of fractional Brownian motion indexed by points of the unit circle of  $\mathbf{R}^2$ , has  $m$ -stationary  $\mathcal{C}_0$ -increments. By definition,  $\mathcal{C}_0$  consists of all elements  $\widehat{MM'}$  where  $M, M' \in \mathbb{S}_1$ . Then this stationarity property states that the law of the process  $\Delta X$  is invariant by translations along  $\mathbb{S}_1$ .*

The following proposition shows that definition 5.1 provides a natural extension of stationarity property for one-parameter processes. Then, it justifies this new definition for stationarity of set-indexed processes.

**Proposition 5.4.** *A set-indexed process  $X = \{X_U; U \in \mathcal{A}\}$  has  $m$ -stationary  $\mathcal{C}_0$ -increments if and only if the  $m$ -standard projection of  $X$  on any elementary flow  $f : \mathbf{R}_+ \rightarrow \mathcal{A}$  has stationary increments, i. e.*

$$\left\{ X_{t+h}^{f,m} - X_h^{f,m}; t \in \mathbf{R}_+ \right\} \stackrel{(d)}{=} \left\{ X_t^{f,m} - X_0^{f,m}; t \in \mathbf{R}_+ \right\}$$

where  $X^{f,m} = \{X_{f \circ \theta^{-1}(t)}; t \in \mathbf{R}_+\}$  and  $\theta : t \mapsto m[f(t)]$ .

*Proof.* We prove the first implication. Assume that  $X$  has  $m$ -stationary  $\mathcal{C}_0$ -increments and that  $f$  is an elementary flow.

For all  $t_1 < t_2 < \dots < t_n$  and  $h$  in  $\mathbf{R}_+$ , consider for  $1 \leq i \leq n$ ,  $U_i = f \circ \theta^{-1}(t_i + h)$ ,  $V = f \circ \theta^{-1}(h)$  and

$$C_i = U_i \setminus V = f \circ \theta^{-1}(t_i + h) \setminus f \circ \theta^{-1}(h) \quad \text{and} \quad A_i = f \circ \theta^{-1}(t_i).$$

As  $(U_i)_{1 \leq i \leq n}$  and  $(A_i)_{1 \leq i \leq n}$  are increasing and  $V \subset U_i$  ( $\forall i$ ), we have

$$\begin{aligned} \Delta X_{U_i \setminus V} &= X_{U_i} - X_V \\ &= X_{f \circ \theta^{-1}(t_i + h)} - X_{f \circ \theta^{-1}(h)} \\ &= X_{t_i + h}^{f,m} - X_h^{f,m} \end{aligned}$$

and

$$\begin{aligned} m(U_i \setminus V) &= m[f \circ \theta^{-1}(t_i + h)] - m[f \circ \theta^{-1}(h)] \\ &= \theta \circ \theta^{-1}(t_i + h) - \theta \circ \theta^{-1}(h) \\ &= t_i \\ &= m(A_i). \end{aligned}$$

Then,  $m$ -stationarity of the set-indexed process  $X$  implies

$$(\Delta X_{U_1 \setminus V}, \dots, \Delta X_{U_n \setminus V}) \stackrel{(d)}{=} (X_{A_1}, \dots, X_{A_n})$$

which gives

$$(X_{t_1+h}^{f,m} - X_h^{f,m}, \dots, X_{t_n+h}^{f,m} - X_h^{f,m}) \stackrel{(d)}{=} (X_{t_1}^{f,m}, \dots, X_{t_n}^{f,m})$$

and the increment stationarity of the  $m$ -standard projection of  $X$  on  $f$ .

In the same way, using lemma 3.3 to define a flow passing through every  $U_i$  ( $1 \leq i \leq n$ ), we prove the converse.  $\square$

**5.2. Self-similarity.** In [HeMe06a], we defined the self-similarity property of a set-indexed process with respect to action of a group  $G$  satisfying the following assumptions.

We suppose that  $\mathcal{A}$  is provided with the operation of a non trivial group  $G$  that can be extended satisfying

$$\begin{aligned} \forall U, V \in \mathcal{A}, \forall g \in G; \quad &g.(U \cup V) = g.U \cup g.V \\ &g.(U \setminus V) = g.U \setminus g.V \end{aligned} \tag{15}$$

and assume there exists a surjective function  $\mu : G \rightarrow \mathbf{R}_+^*$

$$\forall U \in \mathcal{A}, \forall g \in G; \quad m(g.U) = \mu(g).m(U). \quad (16)$$

A set-indexed process  $X = \{X_U; U \in \mathcal{A}\}$  is said to be self-similar of index  $H$ , if there exists a group  $G$  which operates on  $\mathcal{A}$ , and satisfies (15) and (16), such that for all  $g \in G$ ,

$$\{X_{g.U}; U \in \mathcal{A}\} \stackrel{(d)}{=} \{\mu(g)^H.X_U; U \in \mathcal{A}\} \quad (17)$$

**Remark 5.5.** In the case of  $\mathcal{A} = \{[0, u]; u \in \mathbf{R}_+^N\} \cup \{\emptyset\}$ , the operation of  $G = \mathbf{R}_+$  defined by

$$\begin{aligned} \mathbf{R}_+ \times \mathcal{A} &\rightarrow \mathcal{A} \\ (a, [0, u]) &\mapsto [0, au] \end{aligned}$$

satisfies assumptions (15) and (16).

On the contrary to stationarity property, self-similarity of a set-indexed process does not imply self-similarity of projections on flows in a natural way. This is essentially due to the fact that there is no connection between zooming in  $\mathcal{A}$  (through operation of  $G$ ) and zooming along a flow (through multiplication by  $\mathbf{R}_+$ ). For instance, in the frame of remark 5.5, if  $[0, u] \in \mathcal{A}$  belongs to an elementary flow  $f$ , i. e. if there exists  $t \in \mathbf{R}_+$  such that  $f(t) = [0, u]$ , the element  $[0, au]$  ( $a > 0$ ) of  $\mathcal{A}$  does not belong necessarily to  $f$ .

However, under some additional assumptions either about flows or about the set-indexed process, standard projections can inherit the self-similarity property.

**Proposition 5.6.** Let  $X = \{X_U; U \in \mathcal{A}\}$  be a set-indexed process on  $(\mathcal{T}, \mathcal{A}, m)$  which satisfies the two following properties:

- (1) self-similarity of index  $H$  (with respect to operation of a group  $G$  satisfying assumptions (15) and (16)),
- (2)  $m$ -stationarity of  $\mathcal{C}_0$ -increments.

Then, the  $m$ -standard projection of  $X$  on any elementary flow  $f$  is self-similar of index  $H$ , i. e.

$$\forall a \in \mathbf{R}_+; \quad \left\{ X_{at}^{f,m}; t \in \mathbf{R}_+ \right\} \stackrel{(d)}{=} \left\{ a^H.X_t^{f,m}; t \in \mathbf{R}_+ \right\}$$

where  $X_t^{f,m} = X_{f \circ \theta^{-1}(t)}$  and  $\theta : t \mapsto m[f(t)]$ .

*Proof.* Let  $f$  be any elementary flow,  $a \in \mathbf{R}_+$  and  $t_1 < t_2 < \dots < t_n$  a sequence of elements of  $\mathbf{R}_+$ . For all  $i = 1, \dots, n$ , consider  $U_i = f \circ \theta^{-1}(t_i)$ .

As  $\mu$  is a surjective function, there exists  $g \in G$  such that  $a = \mu(g)$ .

As

$$\begin{aligned} \forall i = 1, \dots, n; \quad m(f \circ \theta^{-1}(at_i)) &= \theta \circ \theta^{-1}(at_i) = at_i \\ &= \mu(g)m(U_i) = m(g.U_i), \end{aligned}$$

by  $m$ -stationarity, we have

$$(X_{g.U_1}, \dots, X_{g.U_n}) \stackrel{(d)}{=} (X_{at_1}^{f,m}, \dots, X_{at_n}^{f,m}) \quad (18)$$

and by self-similarity,

$$(X_{g.U_1}, \dots, X_{g.U_n}) \stackrel{(d)}{=} (\mu(g)^H X_{U_1}, \dots, \mu(g)^H X_{U_n}). \quad (19)$$



The result follows from (18) and (19).  $\square$

In the previous proof, the stationarity allows to guarantee for any flow  $f$  and  $U \in \mathcal{A}$  the existence of  $g \in G$  such that  $g.U$  belongs to  $f$ , up to equality with respect to the law of  $X$ . In that context, the  $m$ -stationarity definition allowing deformation of objects in  $\mathcal{A}$  is the key of its special importance.

**Remark 5.7.** *The particular case of set-indexed fractional Brownian motion, which satisfies both properties (1) and (2) of proposition 5.6, shows that projections of  $\text{sifBm}$  on any elementary flow is a self-similar one-parameter process. Of course, this fact is already known as  $m$ -standard projections of  $\text{sifBm}$  are  $f\text{Bm}$ .*

## 6. CHARACTERIZATION OF THE $\text{sifBm}$ BY STATIONARITY AND SELF-SIMILARITY

Real-parameter fractional Brownian motion is well known as the only Gaussian process satisfying the two properties of self-similarity and increment stationarity.

In [HeMe06a], we proved that a set-indexed process  $X = \{X_U; U \in \mathcal{A}\}$  satisfying the two following properties:

- (i)  $\mathcal{C}_0$ -increment stationarity (property (14))
- (ii) self-similarity of index  $H \in (0, 1/2]$

must verify, for all  $U$  and  $V$  in  $\mathcal{A}$  such that  $U \subset V$

$$E[X_U X_V] = K \cdot [m(U)^{2H} + m(V)^{2H} - m(V \setminus U)^{2H}]$$

where  $K \in \mathbf{R}_+$ .

Characterizing covariance between only comparable elements of  $\mathcal{A}$ , the two fractal properties (i) and (ii) only provide a pseudo-characterization of the  $\text{sifBm}$ .

Here, we prove that using the new definition 5.1 of stationarity for a set-indexed process, we get a complete characterization of the  $\text{sifBm}$ . As we see in the proof, the statement which only consider distributional properties of set-indexed processes, relies on the characterization of the  $\text{sifBm}$  by its projections on flows (see theorem 3.4).

**Theorem 6.1.** *The  $\text{sifBm}$   $\mathbf{B}^H$  on  $(\mathcal{T}, \mathcal{A}, m)$  is the only  $L^2$ -monotone outer-continuous Gaussian set-indexed process, which is self-similar of index  $H \in (0, 1/2]$  and has  $m$ -stationary  $\mathcal{C}_0$ -increments.*

*Proof.* From [HeMe06a] and proposition 5.2, we know that the  $\text{sifBm}$  is Gaussian, self-similar and has  $m$ -stationary  $\mathcal{C}_0$ -increments.

Conversely, consider a Gaussian set-indexed process  $X$ , which is self-similar and has  $m$ -stationary  $\mathcal{C}_0$ -increments. For any elementary flow  $f : [a, b] \subset \mathbf{R}_+ \rightarrow \mathcal{A}$ , propositions 5.4 and 5.6 imply that the standard projection of  $X$  on  $f$  satisfies

- $X^{f,m}$  is Gaussian,
- $X^{f,m}$  is self-similar of index  $H$ ,
- $X^{f,m}$  has stationary increments.

Therefore, by the well-known characterization of one-parameter  $f\text{Bm}$ ,  $X^{f,m}$  is a fractional Brownian motion, and then

$$\forall t \in [a, b]; \quad E[X_{f \circ \theta^{-1}(t)}]^2 = t^{2H} \quad (20)$$

where  $\theta : t \mapsto m[f(t)]$ .

Then, theorem 3.4 states the existence of a Borel measure  $\nu$  on  $\mathcal{T}$  such that  $X$  is a sifBm on  $(\mathcal{T}, \mathcal{A}, \nu)$ . Consequently, the centered Gaussian process  $X$  is defined by

$$\forall U, V \in \mathcal{A}; \quad E[X_U - X_V]^2 = \nu(U \triangle V)^{2H},$$

and particularly

$$\forall U \in \mathcal{A}; \quad E[X_U]^2 = \nu(U)^{2H}.$$

Then, according to proposition 2.4, for any elementary flow  $f : [a, b] \rightarrow \mathcal{A}$ , the process  $\{X_{f \circ \psi^{-1}(t)}; t \in [a, b]\}$  with  $\psi : t \mapsto \nu[f(t)]$  is a one-parameter fractional Brownian motion. This leads to

$$\forall t \in [a, b]; \quad E[X_{f \circ \psi^{-1}(t)}]^2 = t^{2H}. \quad (21)$$

From (20) and (21), we get for any elementary flow  $f : [a, b] \rightarrow \mathcal{A}$

$$\forall t \in [a, b]; \quad E[X_{f(t)}]^2 = m[f(t)]^{2H} = \nu[f(t)]^{2H}.$$

Considering a flow passing through any given  $U \in \mathcal{A}$ , this implies

$$\forall U \in \mathcal{A}; \quad m(U) = \nu(U)$$

and consequently, the set-indexed process  $X$  is a sifBm on  $(\mathcal{T}, \mathcal{A}, m)$ .

□

**Example 6.2.** *Let us come back to example 4.5. With the same notation, the sifBm on  $(\mathbb{S}_1, \mathcal{A}, m)$  is the only mean-zero Gaussian process  $X$  indexed by  $\mathbb{S}_1$  such that the two following conditions are satisfied:*

- (i) *the law of the increment process  $\Delta X$  is invariant against translations along the circle;*
- (ii)  *$X$  is self-similar of parameter  $H$ , i.e.*

$$\forall a > 0; \quad \{X_{a.M}; M \in \mathbb{S}_1\} \stackrel{(d)}{=} \{a^H.X_M; M \in \mathbb{S}_1\},$$

where  $a.M$  denotes the point  $M'$  of  $\mathbb{S}_1$  defined by  $m(\widehat{0M'}) = a.m(\widehat{0M})$ .

**Remark 6.3.** *In the view of theorem 6.1, it is natural to wonder about existence of Gaussian set-indexed processes which are self-similar of index  $H \in (1/2, 1)$  and have  $m$ -stationary  $\mathcal{C}_0$ -increments. According to theorem 3.4, this question is related to the existence of set-indexed processes whose standard projections on any flow are fBm of parameter  $H \in (1/2, 1)$ . As we saw in section 4, the answer depends on the structure of the indexing collection  $\mathcal{A}$ .*

#### ACKNOWLEDGMENT

Erick Herbin would like to thank Ely Merzbach for his two kind invitations at Bar Ilan University, where most of this work was done.

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